

15. ESERCIZI

SULL'INTEGRAZIONE PER PARTI

- 1) $\int x \cos x \, dx$ 2) $\int x e^x \, dx$ 3) $\int x \ln x \, dx$ 4) $\int x^3 \ln x \, dx$ 5) $\int \operatorname{arctg} x \, dx$
 6) $\int x^2 \operatorname{arctg} x \, dx$ 7) $\int x^2 e^{3x} \, dx$ 8) $\int \ln^2 x \, dx$ 9) $\int x \sqrt{x+1} \, dx$

Col “circolo vizioso apparente”:

- 10) $\int e^x \sin x \, dx$
 11) $\int e^{2x} \cos x \, dx$
 12) $\int e^{-x} \cos 3x \, dx$

SULL'INTEGRAZIONE PER SOSTITUZIONE

(per questi esercizi più impegnativi, nelle “risposte” sono riportati gli svolgimenti completi)

- 13) Fai vedere che $\int \frac{1}{x\sqrt{x-1}} \, dx = 2 \operatorname{arctg} \sqrt{x-1} + c$, con la sostituzione $\sqrt{x-1} = t$
 14) Fai vedere che $\int x\sqrt{x-1} \, dx = \frac{(x-1)(6x+4)}{15} \sqrt{x-1} + c$, con la sostituzione $\sqrt{x-1} = t$
 15) Dimostra che $\int \sqrt{4-x^2} \, dx = 2 \operatorname{arc} \sin \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2} + c$, ponendo $x = 2 \sin t$
 16a) Dimostra che $\int \frac{1}{x(\ln x + 2)} \, dx = \ln |\ln x + 2| + c$, con la sostituzione $x = e^t$
 16b) L'integrale precedente avrebbe potuto essere ricavato anche senza sostituzioni: in che modo?
 17) Mostra che $\int \frac{\sqrt{x+1}}{x} \, dx = 2\sqrt{x+1} + \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + c$, ponendo $\sqrt{x+1} = t$
 18) Ricava l'uguaglianza $\int \frac{\sqrt{x}}{1-\sqrt{x}} \, dx = -x - 2\sqrt{x} - 2 \ln |\sqrt{x}-1| + c$ tramite la sostituzione $\sqrt{x} = t$
 19) Ponendo $x = \sin t$ ricava l'uguaglianza $\int \sqrt{\frac{1-x}{1+x}} \, dx = \operatorname{arc} \sin x + \sqrt{1-x^2} + c$
 20a) $\int x\sqrt{1+x^2} \, dx = \frac{(1+x^2)\sqrt{1+x^2}}{3} + c$ con la sostituzione $\sqrt{1+x^2} = t$
 20b) $\int x\sqrt{1+x^2} \, dx = \frac{(1+x^2)\sqrt{1+x^2}}{3} + c$ senza alcuna sostituzione
 21) Il laborioso integrale $\int \frac{1}{(1+x^2)^2} \, dx = \operatorname{arctg} x + \frac{x}{1+x^2} + c$ può essere ricavato in due modi:
 a) con la sostituzione $x = \operatorname{tg} t$
 b) coi passaggi $\int \frac{1}{(1+x^2)^2} \, dx = \int \frac{1+x^2-x^2}{(1+x^2)^2} \, dx = \int \left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2} \right) \, dx = \operatorname{arctg} x - \int \frac{x^2}{(1+x^2)^2} \, dx$
 eseguendo l'ultimo integrale per parti: $\int \frac{x^2}{(1+x^2)^2} \, dx = \int \frac{1}{2} x \cdot \frac{2x}{(1+x^2)^2} \, dx = \dots$

Provaci!

RISPOSTE

$$1) x \operatorname{sen} x + \cos x + c \quad 2) x e^x - e^x + c \quad 3) \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

$$4) \frac{x^4}{4} \ln x - \frac{x^4}{16} + c \quad 5) x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + c \quad 6) \frac{x^3}{3} \operatorname{arctg} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + c$$

$$7) \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c \quad 8) x \ln^2 x - 2x \ln x + 2x + c \quad 9) \frac{2(x+1)(3x-2)}{15} \sqrt{x+1} + c$$

$$10) \frac{e^x (\operatorname{sen} x - \cos x)}{2} + c \quad 11) \frac{e^{2x} (2 \cos x + \operatorname{sen} x)}{5} + c \quad 12) \frac{e^{-x} (3 \operatorname{sen} 3x - \cos 3x)}{10} + c$$

13)

$$\int \frac{1}{x\sqrt{x-1}} dx$$

$$\sqrt{x-1} = t$$

$$x-1 = t^2 \rightarrow x = 1+t^2$$

$$dx = 2t dt$$

$$\int \frac{1}{x\sqrt{x-1}} dx = \int \frac{1}{(1+t^2) \cdot t} \cdot 2t dt = \int \frac{2}{1+t^2} dt = 2 \operatorname{arctg} t + c = 2 \operatorname{arctg} \sqrt{x-1} + c$$

14)

$$\int x\sqrt{x-1} dx$$

$$\sqrt{x-1} = t$$

$$x-1 = t^2$$

$$x = 1+t^2$$

$$dx = 2t dt$$

$$\int x\sqrt{x-1} dx = \int (1+t^2) \cdot t \cdot 2t dt = \int (2t^2 + 2t^4) dt =$$

$$= 2 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^5}{5} + c = \frac{2}{3} (\sqrt{x-1})^3 + \frac{2}{5} (\sqrt{x-1})^5 + c = \frac{2}{3} (x-1) \sqrt{x-1} + \frac{2}{5} (x-1)^2 \sqrt{x-1} + c =$$

$$= (x-1) \sqrt{x-1} \left[\frac{2}{3} + \frac{2}{5} (x-1) \right] + c =$$

$$= (x-1) \sqrt{x-1} \cdot \frac{10+6x-6}{15} + c = \frac{(x-1)(6x+4)}{15} \sqrt{x-1} + c$$

15)

$$\int \sqrt{4-x^2} dx$$

$$x = 2 \operatorname{sen} t \rightarrow \operatorname{sen} t = \frac{x}{2}, t = \operatorname{arc} \operatorname{sen} \frac{x}{2}; \cos t = \sqrt{1 - \left(\frac{x}{2}\right)^2} = \sqrt{1 - \frac{x^2}{4}}$$

$$dx = 2 \cos t dt$$

$$\int \sqrt{4-x^2} dx = \int \sqrt{4-4\operatorname{sen}^2 t} \cdot 2 \cos t dt = \int \sqrt{4(1-\operatorname{sen}^2 t)} \cdot 2 \cos t dt = \int 2\sqrt{1-\operatorname{sen}^2 t} \cdot 2 \cos t dt =$$

$$= \int 2\sqrt{\cos^2 t} \cdot 2 \cos t dt =$$

$$= \int 2 \cos t \cdot 2 \cos t dt = \int 4 \cos^2 t dt = 4 \cdot \frac{t + \operatorname{sen} t \cos t}{2} + c =$$

$$= 2t + 2 \operatorname{sen} t \cos t + c = 2 \operatorname{arc} \operatorname{sen} \frac{x}{2} + 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \frac{x^2}{4}} + c = 2 \operatorname{arc} \operatorname{sen} \frac{x}{2} + x \sqrt{\frac{4-x^2}{4}} + c = 2 \operatorname{arc} \operatorname{sen} \frac{x}{2} + \frac{1}{2} x \sqrt{4-x^2} + c$$

$$16a) \int \frac{1}{x(\ln x + 2)} dx$$

$$\ln x = t \rightarrow x = e^t$$

$$dx = e^t dt$$

$$\int \frac{1}{x(\ln x + 2)} dx = \int \frac{1}{\cancel{e^t}(t+2)} \cdot \cancel{e^t} dt =$$

$$= \ln|t+2| + c = \ln|\ln x + 2| + c$$

$$16b) \int \frac{1}{\ln x + 2} \cdot \frac{1}{x} dx$$

$$\int \frac{1}{f(x)} \cdot f'(x) dx$$

$$\ln f(x) + c$$

17)

$$\int \frac{\sqrt{x+1}}{x} dx \quad \sqrt{x+1} = t; \quad x+1 = t^2; \quad x = t^2 - 1; \quad dx = 2t dt$$

$$\begin{aligned} \int \frac{\sqrt{x+1}}{x} dx &= \int \frac{t}{t^2-1} \cdot 2t dt = 2 \int \frac{t^2}{t^2-1} dt = 2 \int \frac{t^2-1+1}{t^2-1} dt = 2 \int \left(1 + \frac{1}{t^2-1} \right) dt = 2t + \cancel{2} \cdot \frac{1}{\cancel{2}} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \\ &= 2t + \ln|t-1| - \ln|t+1| + c = 2\sqrt{x+1} + \ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1| + c = 2\sqrt{x+1} + \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + c \end{aligned}$$

18)

$$\int \frac{\sqrt{x}}{1-\sqrt{x}} dx \quad \sqrt{x} = t; \quad x = t^2; \quad dx = 2t dt$$

$$\begin{aligned} \int \frac{\sqrt{x}}{1-\sqrt{x}} dx &= \int \frac{t}{1-t} \cdot 2t dt = \int \frac{2t^2}{1-t} dt = -2 \int \frac{t^2}{t-1} dt = -2 \int \frac{t^2-1+1}{t-1} dt = -2 \int \left(\frac{(t+1)\cancel{(t-1)}}{\cancel{t-1}} + \frac{1}{t-1} \right) dt = \\ &= -2 \left(\frac{t^2}{2} + t + \ln|t-1| \right) + c = -2 \left(\frac{x}{2} + \sqrt{x} + \ln|\sqrt{x}-1| \right) + c = -x - 2\sqrt{x} - 2\ln|\sqrt{x}-1| + c \end{aligned}$$

19)

$$\int \sqrt{\frac{1-x}{1+x}} dx$$

$$x = \sin t \rightarrow t = \arcsin x, \quad dx = \cos t dt$$

$$\begin{aligned} \int \sqrt{\frac{1-x}{1+x}} dx &= \int \sqrt{\frac{1-\sin t}{1+\sin t}} \cos t dt = \int \sqrt{\frac{(1-\sin t)^2}{(1+\sin t)(1-\sin t)}} \cos t dt = \int \frac{1-\sin t}{\sqrt{1-\sin^2 t}} \cos t dt = \\ &= \int \frac{1-\sin t}{\sqrt{\cos^2 t}} \cos t dt = \int \frac{1-\sin t}{\cancel{\cos t}} \cancel{\cos t} dt = t + \cos t + c = \arcsin x + \cos(\arcsin x) + c = \arcsin x + \sqrt{1-x^2} + c \end{aligned}$$

20a)

$$\int x\sqrt{1+x^2} dx$$

$$\sqrt{1+x^2} = t$$

$$1+x^2 = t^2$$

$$x^2 = t^2 - 1 \rightarrow x = \sqrt{t^2 - 1}$$

$$dx = \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2-1}} dt$$

$$\int x\sqrt{1+x^2} dx = \int \sqrt{\cancel{t^2-1}} \cdot t \cdot \frac{t}{\cancel{\sqrt{t^2-1}}} dt =$$

$$= \int t^2 dt = \frac{t^3}{3} + c = \frac{(\sqrt{1+x^2})^3}{3} + c = \frac{(1+x^2)\sqrt{1+x^2}}{3} + c$$

20b)

$$\int x\sqrt{1+x^2} dx = \frac{1}{2} \int 2x\sqrt{1+x^2} dx$$

$$\frac{1}{2} \int f'(x) \sqrt{f(x)} dx =$$

$$= \frac{1}{2} \int f'(x) [f(x)]^{\frac{1}{2}} dx =$$

$$= \frac{1}{2} \cdot \frac{[f(x)]^{\frac{3}{2}}}{\frac{3}{2}} + c =$$

$$= \frac{1}{3} [f(x)]^{\frac{3}{2}} + c$$

21a)

$$\int \frac{1}{(1+x^2)^2} dx$$

$$x = \operatorname{tg} t = \frac{\operatorname{sent}}{\cos t} = \frac{\operatorname{sent}}{\sqrt{1-\operatorname{sen}^2 t}}$$

$$x^2 = \frac{\operatorname{sen}^2 t}{1-\operatorname{sen}^2 t}; \quad x^2 - x^2 \operatorname{sen}^2 t = \operatorname{sen}^2 t; \quad x^2 = x^2 \operatorname{sen}^2 t + \operatorname{sen}^2 t; \quad x^2 = \operatorname{sen}^2 t (x^2 + 1);$$

$$\operatorname{sen}^2 t = \frac{x^2}{1+x^2}; \quad \operatorname{sent} = \frac{x}{\sqrt{1+x^2}}$$

$$dx = \frac{1}{\cos^2 t} dt$$

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{1}{(1+\operatorname{tg}^2 t)^2} \cdot \frac{1}{\cos^2 t} dt = \int \frac{1}{\left(1+\frac{\operatorname{sen}^2 t}{\cos^2 t}\right)^2} \cdot \frac{1}{\cos^2 t} dt = \int \frac{1}{\left(\frac{\cos^2 t + \operatorname{sen}^2 t}{\cos^2 t}\right)^2} \cdot \frac{1}{\cos^2 t} dt =$$

$$= \int \frac{1}{\left(\frac{1}{\cos^2 t}\right)^2} \cdot \frac{1}{\cos^2 t} dt = \int \cos^4 t \cdot \frac{1}{\cos^2 t} dt = \int \cos^2 t dt = \frac{t + \operatorname{sent} \cos t}{2} + c = \frac{\operatorname{arctg} x + \frac{x}{\sqrt{1+x^2}} \cdot \sqrt{1-\frac{x^2}{1+x^2}}}{2} + c =$$

$$= \frac{\operatorname{arctg} x + \frac{x}{\sqrt{1+x^2}} \cdot \sqrt{\frac{1+x^2-x^2}{1+x^2}}}{2} + c = \frac{\operatorname{arctg} x + \frac{x}{\sqrt{1+x^2}} \cdot \sqrt{\frac{1}{1+x^2}}}{2} + c = \frac{\operatorname{arctg} x + \frac{x}{1+x^2}}{2} + c =$$

$$= \frac{1}{2} \left(\operatorname{arctg} x + \frac{x}{1+x^2} \right) + c$$

21b)

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{1+x^2-x^2}{(1+x^2)^2} dx = \int \left[\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2} \right] dx = \operatorname{arctg} x - \int \frac{x^2}{(1+x^2)^2} dx$$

$$\int \frac{x^2}{(1+x^2)^2} dx = \int \frac{1}{2} x \cdot \frac{2x}{(1+x^2)^2} dx = \frac{1}{2} \int x \cdot \frac{2x}{(1+x^2)^2} dx = \frac{1}{2} \left[x \cdot \left(-\frac{1}{1+x^2} \right) - \int \left(-\frac{1}{1+x^2} \right) \cdot 1 dx \right] = -\frac{1}{2} \cdot \frac{x}{1+x^2} + \frac{1}{2} \operatorname{arctg} x + c$$

$$\int \frac{1}{(1+x^2)^2} dx = \operatorname{arctg} x + \frac{1}{2} \cdot \frac{x}{1+x^2} - \frac{1}{2} \operatorname{arctg} x + c = \frac{1}{2} \operatorname{arctg} x + \frac{1}{2} \cdot \frac{x}{1+x^2} + c$$

