## 15. ESERCIZI

### SULL'INTEGRAZIONE PER PARTI

1) $\int x\cos x\,dx$ 2) $\int xe^x\,dx$ 3) $\int x\ln x\,dx$ 4) $\int x^3\ln x\,dx$ 5) $\int arctg\,x\,dx$

6) $\int x^2\,arctg\,x\,dx$ 7) $\int x^2 e^{3x}\,dx$ 8) $\int \ln^2 x\,dx$ 9) $\int x\sqrt{x+1}\,dx$

Col "circolo vizioso apparente":

10) $\int e^x\,sen\,x\,dx$

11) $\int e^{2x}\cos x\,dx$

12) $\int e^{-x}\cos 3x\,dx$

### SULL'INTEGRAZIONE PER SOSTITUZIONE

(per questi esercizi più impegnativi, nelle "risposte" sono riportati gli svolgimenti completi)

13) Fai vedere che $\int \frac{1}{x\sqrt{x-1}}\,dx = 2\,arctg\sqrt{x-1}+c$, con la sostituzione $\sqrt{x-1}=t$

14) Fai vedere che $\int x\sqrt{x-1}\,dx = \frac{(x-1)(6x+4)}{15}\sqrt{x-1}+c$, con la sostituzione $\sqrt{x-1}=t$

15) Dimostra che $\int \sqrt{4-x^2}\,dx = 2\,arc\,sen\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2}+c$, ponendo $x=2\,sen\,t$

16a) Dimostra che $\int \frac{1}{x(\ln x+2)}\,dx = \ln\left|\ln x+2\right|+c$, con la sostituzione $x=e^t$

16b) L'integrale precedente avrebbe potuto essere ricavato anche senza sostituzioni: in che modo?

17) Mostra che $\int \frac{\sqrt{x+1}}{x}\,dx = 2\sqrt{x+1}+\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+c$, ponendo $\sqrt{x+1}=t$

18) Ricava l'uguaglianza $\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx = -x-2\sqrt{x}-2\ln\left|\sqrt{x}-1\right|+c$ tramite la sostituzione $\sqrt{x}=t$

19) Ponendo $x=sen\,t$ ricava l'uguaglianza $\int \sqrt{\frac{1-x}{1+x}}\,dx = arc\,sen\,x+\sqrt{1-x^2}+c$

20a) $\int x\sqrt{1+x^2}\,dx = \frac{(1+x^2)\sqrt{1+x^2}}{3}+c$ con la sostituzione $\sqrt{1+x^2}=t$

20b) $\int x\sqrt{1+x^2}\,dx = \frac{(1+x^2)\sqrt{1+x^2}}{3}+c$ senza alcuna sostituzione

21) Il laborioso integrale $\int \frac{1}{(1+x^2)^2}\,dx = arctg\,x+\frac{x}{1+x^2}+c$ può essere ricavato in due modi:

a) con la sostituzione $x=tg\,t$

b) coi passaggi $\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx = \int\left(\frac{1}{1+x^2}-\frac{x^2}{(1+x^2)^2}\right)dx = arctg\,x-\int \frac{x^2}{(1+x^2)^2}\,dx$

eseguendo l'ultimo integrale per parti: $\int \frac{x^2}{(1+x^2)^2}\,dx = \int \frac{1}{2}x\cdot\frac{2x}{(1+x^2)^2}\,dx = \ldots$

Provaci!

**RISPOSTE**

1) $x\,sen\,x+\cos x+c$ 2) $xe^x-e^x+c$ 3) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$

4) $\frac{x^4}{4}\ln x-\frac{x^4}{16}+c$ 5) $x\,arctg\,x-\frac{1}{2}\ln\left(1+x^2\right)+c$ 6) $\frac{x^3}{3}arctg\,x-\frac{x^2}{6}+\frac{1}{6}\ln\left(1+x^2\right)+c$

7) $\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+c$ 8) $x\ln^2 x-2x\ln x+2x+c$ 9) $\frac{2(x+1)(3x-2)}{15}\sqrt{x+1}+c$

10) $\frac{e^x(sen\,x-\cos x)}{2}+c$ 11) $\frac{e^{2x}(2\cos x+sen\,x)}{5}+c$ 12) $\frac{e^{-x}(3sen\,3x-\cos 3x)}{10}+c$

13)

$$\int\frac{1}{x\sqrt{x-1}}\,dx$$

$\sqrt{x-1}=t$

$x-1=t^2 \rightarrow x=1+t^2$

$dx=2t\,dt$

$$\int\frac{1}{x\sqrt{x-1}}\,dx=\int\frac{1}{\left(1+t^2\right)\cdot \not t}\cdot 2\not t\,dt=\int\frac{2}{1+t^2}dt=2\,arctg\,t+c=2\,arctg\sqrt{x-1}+c$$

14)

$$\int x\sqrt{x-1}\,dx$$

$\sqrt{x-1}=t$

$x-1=t^2$

$x=1+t^2$

$dx=2t\,dt$

$$\int x\sqrt{x-1}\,dx=\int\left(1+t^2\right)\cdot t\cdot 2t\,dt=\int\left(2t^2+2t^4\right)dt=$$

$$=2\cdot\frac{t^3}{3}+2\cdot\frac{t^5}{5}+c=\frac{2}{3}\left(\sqrt{x-1}\right)^3+\frac{2}{5}\left(\sqrt{x-1}\right)^5+c=\frac{2}{3}(x-1)\sqrt{x-1}+\frac{2}{5}(x-1)^2\sqrt{x-1}+c=$$

$$=(x-1)\sqrt{x-1}\left[\frac{2}{3}+\frac{2}{5}(x-1)\right]+c=$$

$$=(x-1)\sqrt{x-1}\cdot\frac{10+6x-6}{15}+c=\frac{(x-1)(6x+4)}{15}\sqrt{x-1}+c$$

15)

$$\int\sqrt{4-x^2}\,dx$$

$x=2\,sent \rightarrow sent=\frac{x}{2},\ t=arc\,sen\frac{x}{2};\ \ \cos t=\sqrt{1-\left(\frac{x}{2}\right)^2}=\sqrt{1-\frac{x^2}{4}}$

$dx=2\cos t\,dt$

$$\int\sqrt{4-x^2}\,dx=\int\sqrt{4-4sen^2t}\cdot 2\cos t\,dt=\int\sqrt{4\left(1-sen^2t\right)}\cdot 2\cos t\,dt=\int 2\sqrt{1-sen^2t}\cdot 2\cos t\,dt=$$

$$=\int 2\sqrt{\cos^2 t}\cdot 2\cos t\,dt=$$

$$=\int 2\cos t\cdot 2\cos t\,dt=\int 4\cos^2 t\,dt=4\cdot\frac{t+sent\cos t}{2}+c=$$

$$=2t+2sent\cos t+c=2\,arc\,sen\frac{x}{2}+\not 2\cdot\frac{x}{\not 2}\cdot\sqrt{1-\frac{x^2}{4}}+c=2\,arc\,sen\frac{x}{2}+x\sqrt{\frac{4-x^2}{4}}+c=2\,arc\,sen\frac{x}{2}+\frac{1}{2}x\sqrt{4-x^2}+c$$

16a)

$$\int \frac{1}{x(\ln x+2)}dx$$

$$\ln x = t \to x = e^t$$

$$dx = e^t\,dt$$

$$\int \frac{1}{x(\ln x+2)}dx = \int \frac{1}{\cancel{e^t}\,(t+2)}\cdot \cancel{e^t}\,dt =$$

$$= \ln|t+2| + c = \ln|\ln x+2| + c$$

16b)

$$\int \frac{1}{\ln x+2}\cdot\frac{1}{x}dx$$

$$\int \frac{1}{f(x)}\cdot f'(x)\,dx$$

$$\ln f(x) + c$$

17)

$$\int \frac{\sqrt{x+1}}{x}dx \qquad \sqrt{x+1}=t; \quad x+1=t^2; \quad x=t^2-1; \quad dx=2t\,dt$$

$$\int \frac{\sqrt{x+1}}{x}dx = \int \frac{t}{t^2-1}\cdot 2t\,dt = 2\int \frac{t^2}{t^2-1}dt = 2\int \frac{t^2-1+1}{t^2-1}dt = 2\int\left(1+\frac{1}{t^2-1}\right)dt = 2t+\cancel{2}\cdot\frac{1}{\cancel{2}}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt =$$

$$= 2t+\ln|t-1|-\ln|t+1|+c = 2\sqrt{x+1}+\ln\left|\sqrt{x+1}-1\right|-\ln\left|\sqrt{x+1}+1\right|+c = 2\sqrt{x+1}+\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+c$$

18)

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}dx \qquad \sqrt{x}=t; \quad x=t^2; \quad dx=2t\,dt$$

$$\int \frac{\sqrt{x}}{1-\sqrt{x}}dx = \int \frac{t}{1-t}\cdot 2t\,dt = \int \frac{2t^2}{1-t}dt = -2\int \frac{t^2}{t-1}dt = -2\int \frac{t^2-1+1}{t-1}dt = -2\int\left(\frac{(t+1)\cancel{(t-1)}}{\cancel{t-1}}+\frac{1}{t-1}\right)dt =$$

$$= -2\left(\frac{t^2}{2}+t+\ln|t-1|\right)+c = -2\left(\frac{x}{2}+\sqrt{x}+\ln\left|\sqrt{x}-1\right|\right)+c = -x-2\sqrt{x}-2\ln\left|\sqrt{x}-1\right|+c$$

19)

$$\int \sqrt{\frac{1-x}{1+x}}\,dx$$

$$x = sen\,t \to t = arc\,sen\,x, \quad dx = cos\,t\,dt$$

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \sqrt{\frac{1-sen\,t}{1+sen\,t}}\,cos\,t\,dt = \int \sqrt{\frac{(1-sen\,t)^2}{(1+sen\,t)(1-sen\,t)}}\,cos\,t\,dt = \int \frac{1-sen\,t}{\sqrt{1-sen^2 t}}\,cos\,t\,dt =$$

$$= \int \frac{1-sen\,t}{\sqrt{cos^2 t}}\,cos\,t\,dt = \int \frac{1-sen\,t}{\cancel{cos\,t}}\,\cancel{cos\,t}\,dt = t+cos\,t+c = arc\,sen\,x+cos(arc\,sen\,x)+c = arc\,sen\,x+\sqrt{1-x^2}+c$$

20a)

$$\int x\sqrt{1+x^2}\,dx$$

$$\sqrt{1+x^2} = t$$

$$1+x^2 = t^2$$

$$x^2 = t^2-1 \to x = \sqrt{t^2-1}$$

$$dx = \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2-1}}dt$$

$$\int x\sqrt{1+x^2}\,dx = \int \cancel{\sqrt{t^2-1}}\cdot t\cdot\frac{t}{\cancel{\sqrt{t^2-1}}}dt =$$

$$= \int t^2 dt = \frac{t^3}{3}+c = \frac{\left(\sqrt{1+x^2}\right)^3}{3}+c = \frac{(1+x^2)\sqrt{1+x^2}}{3}+c$$

20b)

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int 2x\sqrt{1+x^2}\,dx$$

$$\frac{1}{2}\int f'(x)\sqrt{f(x)}\,dx =$$

$$= \frac{1}{2}\int f'(x)\left[f(x)\right]^{\frac{1}{2}}dx =$$

$$= \frac{1}{2}\cdot\frac{\left[f(x)\right]^{\frac{3}{2}}}{\frac{3}{2}}+c =$$

$$= \frac{1}{3}\left[f(x)\right]^{\frac{3}{2}}+c$$

21a)

$$\int \frac{1}{\left(1+x^2\right)^2}dx$$

$$x = tg\,t = \frac{sent}{cos\,t} = \frac{sent}{\sqrt{1-sen^2 t}}$$

$$x^2 = \frac{sen^2 t}{1-sen^2 t};\quad x^2 - x^2 sen^2 t = sen^2 t;\ x^2 = x^2 sen^2 t + sen^2 t;\ x^2 = sen^2 t\left(x^2+1\right);$$

$$sen^2 t = \frac{x^2}{1+x^2};\ sent = \frac{x}{\sqrt{1+x^2}}$$

$$dx = \frac{1}{cos^2 t}dt$$

$$\int \frac{1}{\left(1+x^2\right)^2}dx = \int \frac{1}{\left(1+tg^2 t\right)^2}\cdot\frac{1}{cos^2 t}dt = \int \frac{1}{\left(1+\frac{sen^2 t}{cos^2 t}\right)^2}\cdot\frac{1}{cos^2 t}dt = \int \frac{1}{\left(\frac{cos^2 t + sen^2 t}{cos^2 t}\right)^2}\cdot\frac{1}{cos^2 t}dt =$$

$$= \int \frac{1}{\left(\frac{1}{cos^2 t}\right)^2}\cdot\frac{1}{cos^2 t}dt = \int cos^4 t\cdot\frac{1}{cos^2 t}dt = \int cos^2 t\,dt = \frac{t + sent\,cos\,t}{2} + c = \frac{arctg\,x + \frac{x}{\sqrt{1+x^2}}\cdot\sqrt{1-\frac{x^2}{1+x^2}}}{2} + c =$$

$$= \frac{arctg\,x + \frac{x}{\sqrt{1+x^2}}\cdot\sqrt{\frac{1+x^2-x^2}{1+x^2}}}{2} + c = \frac{arctg\,x + \frac{x}{\sqrt{1+x^2}}\cdot\sqrt{\frac{1}{1+x^2}}}{2} + c = \frac{arctg\,x + \frac{x}{1+x^2}}{2} + c =$$

$$= \frac{1}{2}\left(arctg\,x + \frac{x}{1+x^2}\right) + c$$

21b)

$$\int \frac{1}{\left(1+x^2\right)^2}dx = \int \frac{1+x^2-x^2}{\left(1+x^2\right)^2}dx = \int\left[\frac{1}{1+x^2} - \frac{x^2}{\left(1+x^2\right)^2}\right]dx = arctg\,x - \int \frac{x^2}{\left(1+x^2\right)^2}dx$$

$$\int \frac{x^2}{\left(1+x^2\right)^2}dx = \int \frac{1}{2}x\cdot\frac{2x}{\left(1+x^2\right)^2}dx = \frac{1}{2}\int x\cdot\frac{2x}{\left(1+x^2\right)^2}dx = \frac{1}{2}\left[x\cdot\left(-\frac{1}{1+x^2}\right) - \int\left(-\frac{1}{1+x^2}\right)\cdot 1dx\right] = -\frac{1}{2}\cdot\frac{x}{1+x^2} + \frac{1}{2}arctg\,x + c$$

$$\int \frac{1}{\left(1+x^2\right)^2}dx = arctg\,x + \frac{1}{2}\cdot\frac{x}{1+x^2} - \frac{1}{2}arctg\,x + c = \frac{1}{2}arctg\,x + \frac{1}{2}\cdot\frac{x}{1+x^2} + c$$